# *The Λ-core model for studying the binding energies of Λ-hypernuclei under pseudo-spin symmetry using the Hellmann potential*

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## Abstract
We have studied the binding energies of a group of single Λ-hypernuclei in a relativistic approach and modeled the single Λ-hypernuclei as a Λ-core binary system. We have selected the Hellmann potential for interaction between the Λ particle and the core. The Dirac equation by using this potential has been solved under the presence of pseudo-spin symmetry in terms of the generalized parametric Nikiforov-Uvarov method. Our results were in agreement with experimental values and other theoretical works. Hence, this model is applicable for the Λ-hypernuclei.
**Keywords: Hypernuclei; Λ-Hyperon; Dirac equation; Pseudo-spin symmetry; Nikiforov-Uvarov method.**

## *Introduction*
The Λ-nucleus interaction is the principal purpose of hypernuclear research [1]. Because of a new degree of freedom, the strangeness, the Λ hyperon does not suffer from Pauli blocking by nucleons of an ordinary nucleus [2, 3]. Hence, the Λ particle can penetrate the nucleus and form strongly bound hypernuclear states, the so-called single Λ-hypernuclei [3].
In this work, we model the single Λ-hypernuclei as a Λ-core binary system like a single nucleon coupled to the whole nuclei to calculate the binding energies of the Λ hyperon. We use the Hellmann potential [4] as the potential between the Λ particle and the core. This type of potential is good enough for defining nucleon-core interaction [5]. Hence, this is suitable for the Λ-core system. Further, it is common to assume the single Λ-hypernuclei as a Λ-core binary system. The binding energies of the ground [6, 7] and first excited [7] states of various single Λ-hypernuclei recently were calculated in the non-relativistic approach by solving the Schrödinger equation for the Λ-core system with the Hulthén potential [8]. In addition, the binding energies of the ground and excited states and the root mean square (RMS) radii of several single Λ-hypernuclei were estimated in relativistic form using Woods–Saxon potential [9] under spin symmetry by hyperon-core model [10].

## *The solution of the Dirac equation*
To characterize relativistic bound states of a spin-1/2 hyperon in the hypernuclei, we consider the wave function of the single hyperon satisfies the time-independent Dirac equation in the following form [11, 12]

$$\left[\hat{\alpha}.c\hat{\mathbf{p}}+\hat{\beta}\left(Mc^2+S(r)\right)\right]\psi_{n_r,\kappa}(r)=\left[E-V(r)\right]\psi_{n_r,\kappa}(r) \quad (1)$$

where $M$, $E$, and $\hat{\mathbf{p}}$ are the single-particle rest mass, total relativistic energy, and the momentum operator, respectively. Furthermore, S(r) and V(r) represent the attractive scalar and repulsive vector potential. The momentum operator and the Dirac matrices are defined as follows:

$$\hat{\mathbf{p}}=-i\hbar\nabla,\quad \hat{\alpha}=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix},\quad \hat{\beta}=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \qquad (2)$$

where $\sigma_i$ are two-dimensional Pauli matrices. Further, $I$ is the identity matrix. The Dirac Hamiltonian in a central potential can commute with the total angular momentum $\hat{j}$ and the spin-orbit coupling operator $\hat{K}=\beta(\sigma.\mathbf{L}+1)$, where $\mathbf{L}$ is the orbital angular momentum. The eigenvalue of the total angular momentum $\hat{j}$ is $j$. The eigenvalues of spin-orbit coupling operator $\hat{K}$ are $\kappa = \pm (j+1/2)$. The positive sign is for the unaligned spin, and the negative sign is for the aligned spin. Thus, we can write the Dirac spinors in terms of the total angular momentum $j$, the spin-orbit quantum number κ, and the radial quantum number $n_r$ in a central field as

$$\psi_{n_r,\kappa}(r,\theta,\phi)=\frac{1}{r}\begin{pmatrix}F_{n_r,\kappa}(r)Y^{l}_{jm}(\theta,\phi)\\ iG_{n_r,\kappa}(r)Y^{\tilde{l}}_{jm}(\theta,\phi)\end{pmatrix}, \qquad (3)$$

where $F_{n_r,\kappa}(r)$ are the upper components, $G_{n_r,\kappa}(r)$ are the lower components of the Dirac spinors. $Y^{l}_{jm}(\theta,\phi)$ and $Y^{\tilde{l}}_{jm}(\theta,\phi)$ are the spherical harmonic functions, in which $m$ is the projection of the angular momentum on the z-axis. In addition, $l$ and $\tilde{l}$ are the orbital angular momentum quantum numbers indicating the spin and pseudo-spin quantum numbers.
Also, we use The Hellmann potential as the potential between the Λ particle and the core that is defined as [13]

$$V(r)=-\frac{a}{r}+\frac{b}{r}e^{-\alpha r} \quad (4)$$

where a and b are assumed as the strengths of the Coulomb and the Yukawa potentials and α as the screening parameter.
To find the energy equation under the pseudo-spin symmetry, we derive the following second-order equation from Eq. (1)

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-\frac{1}{\hbar^2c^2}\left(Mc^2+E-\Delta(r)\right)\left(Mc^2-E\right)\right]G_{n_r,\kappa}(r)=0 \quad (5)$$

by considering [13]

$$\Sigma(r)=V(r)+S(r)=0$$
$$\Delta(r)=V(r)-S(r) \quad . (6)$$
$$S(r)\approx -V(r)$$

we solve it with the Nikiforov-Uvarov method that has been expressed in ref. [11]. So, the energy equation under the pseudo-spin symmetry can be written as Eq. (7)

$$n^2+n+\frac{1}{2}+(2n+1)\left[\sqrt{\frac{1}{4}+\kappa(\kappa-1)}+\sqrt{-\frac{\tilde{\beta}}{\alpha^2}+\frac{2a}{\alpha}\tilde{\gamma}+\kappa(\kappa-1)}\right] \quad (7)$$
$$-\frac{2\tilde{\gamma}}{\alpha}(b-a)+2\kappa(\kappa-1)$$
$$+2\sqrt{\left(\frac{1}{4}+\kappa(\kappa-1)\right)\left(-\frac{\tilde{\beta}}{\alpha^2}+\frac{2a}{\alpha}\tilde{\gamma}+\kappa(\kappa-1)\right)}=0$$

by using the following transformation:

$$\tilde{\beta}=\frac{\left[E^2-M^2c^4\right]}{\hbar^2c^2},\ \tilde{\gamma}=\frac{\left[Mc^2-E\right]}{\hbar^2c^2} \quad . (8)$$

## *Results and discussion*

Our objective is to find the binding energies of the ground and excited bound states of Λ particle by using the relativistic energy equation. Therefore, we apply: $Mc^2$ = 1115 MeV [10]; E=-(-$E_B$+ $Mc^2$) [10], where $E_B$ is the binding energy of the Λ hyperon of the particular state; n=1 [13]; and $\kappa=-\tilde{l}$ [13] to find strengths of Coulomb (a) and the Yukawa (b) potentials in MeV and α as the screening parameter in $fm^{-1}$ by fitting with the experimental results. At last, we calculate binding energies of the 1s and 1p states of the Λ particle in a group of single Λ-hypernuclei and list them in tables 1 and 2. Besides, we calculate the binding energies of Λ in 1d and 1f states for ${}_{\Lambda}^{89}Y$ that can be seen in table 3.

**Table 1**. The ground state binding energy of Λ (MeV)

| Hypernuclei | Coefficients of the potential | | $E_B$-Our | $E_B$-Exp.[14] | $E_B$-Other.[10] | $E_B$-Other.[7] |
|---|---|---|---|---|---|---|
| ${}_{\Lambda}^{13}C$ | α | 0.15 | 11.9197 | 11.69 | 11.41 | 11.447 |
| | a | 0.4096 | | | | |
| | b | -58.6925 | | | | |
| ${}_{\Lambda}^{16}N$ | α | 0.12 | 12.5293 | 13.67 | 11.40 | 12.967 |
| | a | 1.85156 | | | | |
| | b | -57.8448 | | | | |
| ${}_{\Lambda}^{16}O$ | α | 0.12 | 13.2626 | 13.0 | 12.86 | 12.645 |
| | a | 2.1925 | | | | |
| | b | -56.5075 | | | | |
| ${}_{\Lambda}^{28}Si$ | α | 0.05 | 17.5490 | 17.20 | 17.09 | 17.240 |
| | a | 4.3675 | | | | |
| | b | -53.4810 | | | | |
| ${}_{\Lambda}^{32}S$ | α | 0.04 | 17.8397 | 17.50 | 17.33 | - |
| | a | 3.3773 | | | | |
| | b | -53.6510 | | | | |
| ${}_{\Lambda}^{40}Ca$ | α | 0.083 | 18.9076 | 18.70±1.1[15] | 18.39 | 18.493 |
| | a | -45.0112 | | | | |
| | b | -95.5675 | | | | |
| ${}_{\Lambda}^{51}V$ | α | 0.084 | 21.8460 | 21.50 | 21.65 | 18.964 |
| | a | -56.5173 | | | | |
| | b | -108.7796 | | | | |
| ${}_{\Lambda}^{89}Y$ | α | 0.0318 | 23.9859 | 23.60 | 23.69 | 19.928 |
| | a | -102.8435 | | | | |
| | b | -157.9973 | | | | |

**Table 2.** The first excited state binding energy of Λ(MeV)

| Hypernuclei | $E_B$-Our | $E_B$-Exp. [14] | $E_B$-Other [10] | $E_B$-Other.[7] |
|---|---|---|---|---|
| $_{\Lambda}{}^{13}C$ | 0.8083 | 0.8 | 0.85 | 2.24 |
| $_{\Lambda}{}^{16}N$ | 2.2686 | 2.84 | 2.79 | 2.53 |
| $_{\Lambda}{}^{16}O$ | 2.5287 | 2.5 | 2.32 | - |
| $_{\Lambda}{}^{28}Si$ | 7.6734 | 7.6 | 7.68 | 8.543 |
| $_{\Lambda}{}^{32}S$ | 8.3207 | 8.20 | 8.25 | - |
| $_{\Lambda}{}^{40}Ca$ | 11.0832 | 11.00 | 11.06 | 8.138 |
| $_{\Lambda}{}^{51}V$ | 13.4855 | 13.4 | 13.48 | 9.09 |
| $_{\Lambda}{}^{89}Y$ | 15.1186 | 17.7 | 17.53 | 12.834 |

**Table 3.** 1d and 1f binding energies of Λ in $_{\Lambda}{}^{89}Y$(MeV)

| | | $E_B$-Our(MeV) | $E_B$-Exp. [14] | $E_B$-Other.[10] |
|---|---|---|---|---|
| $_{\Lambda}{}^{89}Y$ | 1d | 3.6855 | 3.7 | 3.10 |
| | 1f | 10.9775 | 10.9 | 11.13 |

## *Conclusions*

In this work, we found the energy equation under pseudo-spin symmetry. Then, we calculated the optimum form of Hellmann potential for any of the studied Λ–hypernuclei and the binding energy of Λ for them. Our results are in agreement with the experimental values and other theoretical works. However, there is a notable difference between experimental data and our calculated binding energy in a few cases like the 1p state of $_{\Lambda}{}^{89}Y$. Hence, this model is applicable for the Λ-hypernuclei.

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